Linear Algebra II

11/03/2022, Friday, 09:00 – 11:00

1 $(8+6+6=20 \text{ pts})$ **Inner product spaces**

Let $\mathcal{V}$ be a real inner product space with inner product $\langle u, v\rangle$.

(a) For any $u \in \mathcal{V}$ define $\|u\| := \sqrt{\langle u, u\rangle}$. Prove that $\|\cdot\|$ is a norm on $\mathcal{V}$.

(b) Let u and v be nonzero vectors in $\mathcal{V}$. Determine $a \in \mathbb{R}$ such that the vectors $av - u$ and v are orthogonal. The vector $p := av$ is then called the orthogonal projection of u onto v.

(c) Show that $\|u\|^2 = \|u-p\|^2 + \|p\|^2$.

2 $(5+5+5+5=20 \text{ pts})$ **Least squares approximation**

Let P_3 be the vector space of all real polynomials $p(x)$ with degree less than or equal to 2, i.e.

$$P_3 = \{p(x) \mid p(x) = p_0 + p_1 x + p_2 x^2,\ p_i \in \mathbb{R}\}$$

Let c_1, c_2, c_3 be distinct real numbers. For any pair of polynomials $p(x), q(x) \in P_3$, define $\langle p(x), q(x)\rangle := p(c_1)q(c_1) + p(c_2)q(c_2) + p(c_3)q(c_3)$.

(a) Prove that $\langle\cdot,\cdot\rangle$ defines an inner product on P_3.

(b) Find a condition on the real numbers c_1, c_2, c_3 so that the polynomials 1 and x are orthogonal.

In the remainder of this problem, take $c_1 = 1, c_2 = -1, c_3 = 0$.

(c) Determine an orthonormal basis of the subspace P_2 of P_3.

(d) Compute the orthogonal projection $p(x)$ of the polynomial x^2 onto the subspace P_2.

3 $(5+5+5+5+5 = 25 \text{ pts})$ **Diagonalization**

Consider the "*Laplacian*" matrix A given by

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

(a) Determine the characteristic polynomial of A.

(b) Determine the eigenvalues of A.

(c) Compute corresponding eigenvectors.

(d) Does there exist an orthogonal matrix $Q \in \mathbb{R}^{3\times 3}$ and a diagonal matrix $\Lambda \in \mathbb{R}^{3\times 3}$ such that $A = Q\Lambda Q^\top$? If so, compute such matrices Q and Λ. If not, explain why.

(e) Compute a matrix $B \in \mathbb{R}^{3\times 2}$ such that $A = BB^\top$.

4 $(5+3+3+7+7 = 25 \text{ pts})$ **Unitary matrices**

A matrix $A \in \mathbb{C}^{n\times n}$ is called *skew-Hermitian* if $A^H = -A$.

(a) Show that all eigenvalues of a skew-Hermitian matrix lie on the imaginary axis.

(b) Let A be skew-Hermitian. Show that $I + A$ is nonsingular.

(c) Show that for every $X \in \mathbb{C}^{n\times n}$ we have $(I+X)(I-X) = (I-X)(I+X)$.

(d) Let A be skew-Hermitian. Define the new matrix $M = (I-A)(I+A)^{-1}$. Show that M is unitary.
Hint: a matrix $X \in \mathbb{C}^{n\times n}$ is nonsingular if and only if X^H is nonsingular, and in this case we have: $(X^{-1})^H = (X^H)^{-1}$.

(e) Let $M \in \mathbb{C}^{n\times n}$ be unitary and assume that $I + M$ is nonsingular. Define the new matrix $A = (I-M)(I+M)^{-1}$. Show that A is skew-Hermitian.

10 pts free